Two Liouville-closed Hardy fields

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Let $\mathrm{Li}(\mathbb{R})$ denote the Liouville closure of the field of constants $\mathbb{R}$ in say transseries.

Question 1. Is $\mathrm{Li}(\mathbb{R})$ closed under composition?

Before answering the question in the positive, let me introduce an equivalent definition of Liouville extensions. Given two Hardy fields $\mathcal{H} \subseteq \mathcal{L}$, the extension $\mathcal{L}/\mathcal{H}$ is a Liouville extension if for all $h \in L$, there are an $n \in \mathbb{N}$ and non-zero germs $f_1, \ldots, f_n \in L$ with $h = f_n$, such that for each $i \in \{1, \ldots, n\}$, one of the following occurs:

a) f_i is algeraic over $\mathcal{H}(f_1, \ldots, f_{i-1})$,

b) $f_i' \in \mathcal{H}(f_1, \ldots, f_{i-1})$, or

c) $f_i^\dagger \in \mathcal{H}(f_1, \ldots, f_{i-1})$.

Given $h \in L$, we define $\mathrm{rk}_{\mathcal{H}}(h)$ to be the least $n \in \mathbb{N}$ such that such a decomposition of length n exists. Note that $\mathrm{rk}_{\mathcal{H}}(h) = 0$ if and only if $h \in \mathcal{H}$. Moreover, if $n = \mathrm{rk}_K(h)$ in the situation above, then we have $\mathrm{rk}_K(f_i) < n$ for all $i \in \{1, \ldots, n-1\}$, which is why I use this presentaion of Liouville extensions.

Proposition 1. *The Liouville-closure* $\mathrm{Li}(\mathbb{R})$ *of* $\mathbb{R}$ *is closed under composition.*

Proof. Let us prove by induction on $\mathrm{rk}_{\mathbb{R}}(h)$ that for all $h \in \mathrm{Li}(\mathbb{R})$ and $g \in \mathrm{Li}(\mathbb{R})^{>\mathbb{R}}$, we have $h \circ g \in \mathrm{Li}(\mathbb{R})$. This is immediate when $\mathrm{rk}_{\mathbb{R}}(h) = 0$, since then h is a constant. Let $n \in \mathbb{N}^{>0}$ such that the result holds for germs of rank $< n$ and let $h \in \mathrm{Li}(\mathbb{R})$ with $\mathrm{rk}_{\mathbb{R}}(h) = n$.

There are non-zero germs $f_1, \ldots, f_n \in \mathrm{Li}(\mathbb{R})$ with $h = f_n$, and where for all $i \in \{1, \ldots, n\}$, one of the following occurs:

- f_i is algeraic over $\mathbb{R}(f_1, \ldots, f_{i-1})$,
- $f_i' \in \mathbb{R}(f_1, \ldots, f_{i-1})$,
- $f_i^\dagger \in \mathbb{R}(f_1, \ldots, f_{i-1})$.

Note that for each $i \in \{1, \ldots, n-1\}$, we have $\mathrm{rk}_{\mathbb{R}}(f_i) < n$ whence $f_i \circ g \in \mathrm{Li}(\mathbb{R})$. We distinguish three cases.

Case 1: h is algebraic over $\mathbb{R}(f_1, \ldots, f_{n-1})$. Then $h \circ g$ is algebraic over $\mathbb{R}(f_1 \circ g, \ldots, f_{i-1} \circ g)$, whence algeraic over $\mathrm{Li}(\mathbb{R})$. But $\mathrm{Li}(\mathbb{R})$ is real-closed, so $h \circ g \in \mathrm{Li}(\mathbb{R})$.

Case 2: $h' \in \mathbb{R}(f_1, \ldots, f_{i-1})$. Then

$$(h \circ g)' = g'\, h' \circ g \in g'\, \mathbb{R}(f_1 \circ g, \ldots, f_{i-1} \circ g) \subseteq \mathrm{Li}(\mathbb{R}).$$

Since $\mathrm{Li}(\mathbb{R})$ is closed under integration, it follows that $h \circ g \in \mathrm{Li}(\mathbb{R})$.

Case 3: $h^\dagger \in \mathbb{R}(f_1, \ldots, f_{i-1})$. As in the previous case $(h \circ g)^\dagger = g'\, h^\dagger \circ g \in \mathrm{Li}(\mathbb{R})$. We deduce since $\mathrm{Li}(\mathbb{R})$ is closed under exponential integration that $h \circ g \in \mathrm{Li}(\mathbb{R})$.

This concludes the inductive proof. □

Remark 2. This can be generalized to an arbitrary Hardy field $\mathcal{H}$ closed under composition (instead of $\mathbb{R}$), provided that

$$\mathcal{H} \circ \mathrm{Li}(\mathcal{H}) \subseteq \mathrm{Li}(\mathcal{H}),$$

which of course is a problematic inclusion on its own.

Let E denote Boshernitzan's Hardy field, i.e. E is the intersection of all maximal Hardy fields. In one of your lectures, I asked you if it was known whether every positive infinite germ in E has a level in the sense of Rosenlicht/Marker-Miller. I.e. given $f \in E^{>\mathbb{R}}$, is there an $n \in \mathbb{Z}$ with $\log_k f \asymp \log_{k-n}$ for sufficiently large $k \in \mathbb{N}$?

Proposition 3. *Each element of $\mathcal{B}^{>\mathbb{R}}$ has a level.*

Proof. This can be deduced from a result of Joris in his *Transserial Hardy fields* paper [1]. Consider the field $\mathbb{T}_g$ of grid-based transseries. Let $\mathbb{T}_{\mathrm{da}}$ denote the subfield of $\mathbb{T}_g$ of transseries. This is an $\boldsymbol{\omega}$-free, Newtonian, Liouville-closed H-field with small derivation. By [1, Theorem 5.12], there is a Hardy field $\mathcal{H}$ closed under exp and log and an isomorphism

$$(\mathbb{T}_{\mathrm{da}}, +, \times, <, \prec, \partial, \log) \longrightarrow (\mathcal{H}, +, \times, <, \prec, {}', \log).$$

In particular $\mathcal{H}$ is H-closed. Let $f \in E$ and assume for contradiction that $f \notin \mathcal{H}$. Let $M \supseteq \mathcal{H}$ be a maximal Hardy field containing $\mathcal{H}$. We have $f \in M$ by definition of E. Now $f \in M \setminus \mathcal{H}$ must be d-transcendant over $\mathcal{H}$, hence also over $\mathbb{R}$. This contradicts Boshernitzan's result that each element of E is in fact d-algebraic. Thus $E \subseteq \mathcal{H}$. In particular, the field E embeds into $\mathbb{T}_g$ as an ordered exponential field, so each $f \in E^{>\mathbb{R}}$ has a level $n \in \mathbb{Z}$. $\square$

As far as I know, the only Hardy field with composition which is known not to have levels in this sense is that which is defined in Adele Padgett's forthcoming thesis [2].

Bibliography

[1] J. van der Hoeven. Transserial Hardy fields. *Differential Equations and Singularities. 60 years of J. M. Aroca*, 323:453–487, 2009.

[2] A. Padgett. *Sublogarithmic-transexponential series*. PhD thesis, Berkeley, 2022.